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COHOMOLOGICAL RADON TRANSFORM AND MIXED TWISTED EHRHART POLYNOMIAL

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Let $f_1,\dots,f_r$ be Laurent polynomials in $\mathbf{C}[x_1^{\pm},\dots,x_n^{\pm}]$, $\alpha_1,\dots,\alpha_r$ $\beta_1,\dots,\beta_n$ be rational numbers in $\frac{1}{m}\mathbf{Z}$. For a multivalved function $\eta=f_1^{-\alpha_1}\cdots f_r^{-\alpha_r}x_1^{\beta_1}\cdots x_n^{\beta_n}$, the corresponding local system on $U=\mathbf{C}^N-\{f_1\cdots f_r\neq 0\}$ is denoted by $\mathbf{C}(\eta)$. In this note, we study the Hodge structure of $H_c^i(U,\mathbf{C}(\eta))$. Let $\mathbf{T}(x)$ and $\mathbf{T}(t)$ be n-dimensional tori $\operatorname{Spec}\mathbf{C}[x_1^{\pm},\dots,x_n^{\pm}]$ and $\operatorname{Spec}[t_1^{\pm},\dots,t_n^{\pm}]$ respectively and $\pi:\mathbf{T}(t)\to\mathbf{T}(x)$ be the morphism defined by $x_i=t_i^m$ $(i=1,\dots,n)$. Define finite coverings Y and $\tilde{Y}$ of $\mathbf{T}(x)$ and $\mathbf{T}(t)$ by $Y=Spec[x_1^{\pm},\dots,x_n^{\pm},w_1^{\pm},\dots,w_r^{\pm}]/(w_i^m-f_i)_{i=1,\dots,r}$ and $\tilde{Y}=Y\times_{\mathbf{T}(x)}\mathbf{T}(t)$ respectively. The natural homomorphisms are denoted by $\phi:Y\to\mathbf{T}(x)$ and $\tilde{\phi}:\tilde{Y}\to\mathbf{T}(t)$ respectively. For elements $g=(g_1,\dots,g_r)\in\mu_m^r$ and $h=(h_1,\dots,h_n)\in\mu_m^n$, we define automorphisms of Y and $\mathbf{T}(t)$ over $\mathbf{T}(x)$ by

$$\mu_m^r\ni g\mapsto(w_i\mapsto g_iw_i)\in\operatorname{Aut}(Y/\mathbf{T}(x))$$
$$\mu_m^r\ni h\mapsto(t_j\mapsto h_jt_j)\in\operatorname{Aut}(\mathbf{T}(t)/\mathbf{T}(x))$$

respectively. Via the above homomorphisms, μ_m^r and μ_m^n are identified with subgroups of $\operatorname{Aut}(Y/\mathbf{T}(x))$ and $\operatorname{Aut}(\mathbf{T}(t)/\mathbf{T}(x))$ respectively. The group $\mu_m^r\times\mu_m^n$ is identified with a subgroup of $\operatorname{Aut}(\tilde{Y}/\mathbf{T}(x))$ by fiber product. For characters χ_1 and χ_2 of μ_m^r and μ_m^n, we define $(\phi_!\mathbf{C})(\chi_1^{-1})$, $\pi_!\mathbf{C}(\chi_2)$ and $\tilde{\phi}_!\mathbf{C}(\chi_1^{-1},\chi_2)$ through the actions defined as before. Then we have

$$(\tilde{\phi}_!)(\chi_1^{-1},\chi_2)\simeq(\phi_!\mathbf{C})(\chi_1^{-1})\otimes(\pi_!\mathbf{C})(\chi_2).$$

From now on, we assume the following conditions:

(*) The variety $D_i=\{x\in\mathbf{T}(x)\mid f_i=0\}$ is irreducible and $D_i\neq D_j$.
(**) $\alpha_1,\dots,\alpha_r,\alpha_1+\cdots+\alpha_r$ are not an integers.

Let χ_1,χ_2 be characters corresponding to $(a_1,\dots,a_r)\in(\mathbf{Z}/m)^r$ and $(b_1,\dots,b_n)\in(\mathbf{Z}/m)^n$, where $a_i=m\alpha_i$ and $b_i=m\beta_i$. If we define $Y^0=\phi^{-1}(U)$ and $\phi^0=\phi\mid_{Y^0}$, then under the above conditions (*) and (**), we have $(\phi_!\mathbf{C})(\chi_1^{-1})=j_!(\phi_!^0\mathbf{C})(\chi_1^{-1})$, where j is the natural inclusion $j:U\to\mathbf{T}(x)$. Therefore we have an isomorphism $H_c^i(U,\mathbf{C}(\eta))\simeq H_c^i(\mathbf{T}(x),(\tilde{\phi}_!\mathbf{C})(\chi_1^{-1},\chi_2))$.

We compute $(\phi_! \mathbf{C})(\chi_1^{-1})$ via a family of Fermat hypersurfaces. Let $\mathbf{T}(u) = \operatorname{Spec} \mathbf{C}[u_1^{\pm}, \dots u_r^{\pm}]$ and $\mathcal{Y}$ be a hypersurface of $\mathbf{T}(u) \times \mathbf{T}(x)$ defined by

$$\mathcal{Y} : u_1^m f_1(x) + \cdots + u_r^m f_r(x) = 1.$$

The composite of the natural inclusion $\mathcal{Y} \to \mathbf{T}(u) \times \mathbf{T}(x)$ and the second projection $\mathbf{T}(u) \times \mathbf{T}(x) \to \mathbf{T}(x)$ is denoted by $\psi : \mathcal{Y} \to \mathbf{T}(x)$. This is a family of Fermat hypersurfaces parameterized by $(x_1, \dots, x_n) \in \mathbf{T}(x)$. We define Fermat hypersurface F^{r-1} by $F^{r-1} : \xi_1^m + \cdots + \xi_r^m = 1$ in $\mathbf{T}(\xi) = \operatorname{Spec} \mathbf{C}[\xi_1^{\pm}, \dots, \xi_r^{\pm}]$. The group μ_m^r acts on F^r by

$$\mu_m^r \ni g \mapsto (\xi_i \mapsto g_i \xi_i) \in \operatorname{Aut}(F^{r-1})$$

Through this action, we define $H^{r-1}(F^{r-1}, \mathbf{C})(\chi_1)$ for a character χ_1 of μ_m^r.

Proposition 1. *Under the condition (*) and (**), we have a motivic isomorphism:*

$$(R^i \psi_! \mathbf{C})(\chi_1) \simeq \begin{cases} (\phi_! \mathbf{C})(\chi_1) \otimes H^{r-1}(F^{r-1}, \mathbf{C}) & (i = r-1) \\ 0 & (i \neq r-1). \end{cases}$$

Proof. Let $\mathcal{Y}^0$ be the inverse image $\psi^{-1}(U)$ of U and ψ^0 be the restriction of ψ to $\mathcal{Y}^0$. First we prove

$$(1) \qquad (R^i \psi_!^0 \mathbf{C})(\chi_1) \simeq \begin{cases} (\phi_!^0 \mathbf{C})(\chi_1) \otimes H^{r-1}(F^{r-1}, \mathbf{C}) & (i = r-1) \\ 0 & (i \neq r-1). \end{cases}$$

We define an isomorphism ν between $\mathcal{Y}^0 \times_U Y^0$ and $F^{r-1} \times Y^0$ by

$$\nu : \mathcal{Y}^0 \times_U Y^0 \ni (u_i, w_i) \mapsto (\xi_i = u_i w_i, w_i) \in F^{r-1} \times Y^0.$$

Then if we define a homomorphism

$$\mu : \mu_m^r \times \mu_m^r \ni (g, h) \mapsto (gh, h) \in \mu_m^r \times \mu_m^r,$$

the following diagram commutes.

$$\begin{tikzcd} \mathcal{Y}^0 \times_U \mathcal{Y}^0 \arrow[r, "(g,h)"] \arrow[d, "\nu"'] & \mathcal{Y}^0 \times Y^0 \arrow[d, "\nu"'] \\ F^{r-1} \times Y^0 \arrow[r, "\mu(g,h)"] & F^{r-1} \times Y^0 \end{tikzcd}$$

We denote ψ_1 and ψ_2 the natural morphisms $\psi_1 : \mathcal{Y}^0 \times_U Y^0 \to U$ and $\psi_2 : F^{r-1} \times Y^0 \to U$, respectively. Then we have the following isomorphisms:

$$\begin{aligned} (R^i \psi_!^0 \mathbf{C})(\chi_1) &\simeq (R^i \psi_{1!}^0 \mathbf{C})(\chi_1, 1) \\ &\simeq (R^i \psi_{2!}^0 \mathbf{C})(\chi_1, \chi_1^{-1}) \\ &\simeq H_c^i(F^{r-1}, \mathbf{C})(\chi_1) \otimes (\phi_!^0 \mathbf{C})(\chi_1^{-1}) \end{aligned}$$

By the condition for $a_1, \dots, a_r$, we have $H^i(F^{r-1}, \mathbf{C})(\chi_1) = 0$ if $i \neq r-1$. Therefore we have the identity (1). Since $(\phi_! \mathbf{C})(\chi_1^{-1})_x = 0$ for $x \in D = f_1 \cdots f_r = 0$, it is enough to prove that $(R^i \psi_! \mathbf{C})(\chi_1)_x = 0$ for $x \in D$. But if $f_i(x) = 0$, the $1 \times \cdots \times \overset{i}{\mu_m} \times \cdots \times 1$ acts trivially on the stalk $(R^i \psi_! \mathbf{C})_x$ by proper base change theorem. Therefore we have the required vanishing.

Let $\tilde{\mathcal{Y}}$ be the fiber product $\mathcal{Y} \times_{\mathbf{T}(x)} \mathbf{T}(t)$ and $\mu_m^r \times \mu_m^n$ acts on $\tilde{\mathcal{Y}}$ induced by the actions of μ_m^r and μ_m^n on $\mathcal{Y}$ and $\mathbf{T}(t)$ respectively.

Corollary 2. *Under the same condition of Proposition 1. We have the following isomorphism*

$$H_c^{i+r-1}(\tilde{\mathcal{Y}}, \mathbf{C})(\chi_1, \chi_2) \simeq H_c^i(\mathbf{T}(x), (\tilde{\phi}_!\mathbf{C})(\chi_1^{-1}, \chi_2)) \otimes H^{r-1}(F^{r-1}, \mathbf{C})(\chi_1)$$

Moreover, this isomorphism is compatible with the Hodge filtrations.

Proof. Let $\tilde{\psi} : \tilde{\mathcal{Y}} \to \mathbf{T}(x)$ be the natural morphism. Since

$$\begin{aligned}(R^q\tilde{\psi}_!\mathbf{C})(\chi_1, \chi_2) &\simeq (R^q\psi_!\mathbf{C})(\chi_1) \otimes (\pi_!\mathbf{C})(\chi_2) \\ &\simeq \begin{cases} 0 & (q \neq r-1) \\ (\phi_!\mathbf{C})(\chi_1^{-1}) \otimes (\pi_!\mathbf{C})(\chi_2) \otimes H^{r-1}(F^{r-1}, \mathbf{C})(\chi_1) & (q = r-1), \end{cases}\end{aligned}$$

the spectral sequence

$$E_2^{p,q} = H_c^p(\mathbf{T}(x), (R^q\tilde{\psi}_!\mathbf{C})(\chi_1, \chi_2)) \Rightarrow E_\infty^{p+q} = H_c^{p+q}(\tilde{\mathcal{Y}}, \mathbf{C})(\chi_1, \chi_2)$$

degenerates at E_2-term and we get the theorem.

Remark. So far, we did not assume the resonance condition nor the non-degenerate condition. (See the definitions below.)

Using this identity, we will state the Cohomological interpretation of Radon transform.

Definition. Let $\Delta_1, \ldots, \Delta_r$ be convex polytopes. The polytope $\tilde{\Delta}$ is defined by the convex hull of $(1, 0, \ldots, 0, \Delta_1), \ldots, (0, \ldots, 0, 1, \Delta_r)$ and 0 in $\mathbf{R}^{r+n}$. A sequence of rational number $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_n$ is said to be non resonant if $\partial\tilde{\Delta} \cap (\mathbf{Z}^{r+n} + (\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_n)) = \emptyset$.

If the Newton polygon of $f_1, \ldots, f_r$ is $\Delta_1, \ldots, \Delta_r$ respectively, the Newton polygon of $v_1 f_1 + \cdots + v_r f_r - 1$ is $\tilde{\Delta}$.

Theorem 3 (Cohomological Radon transformation). *Under the same condition of Proposition 1 and assume that*

(1) $u_1^m f_1 + \cdots + u_r^m f_r - 1$ *is non degenerated with respect to its Newton polygon.*
(2) $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_n$ *is non resonant with respect to* $\Delta_1, \ldots, \Delta_r$.

Then $H_c^i(U, \mathbf{C}(\eta)) = 0$ *if* $i \neq n$ *and*

$$H_c^n(U, \mathbf{C}(\eta)) \simeq H_c^{n+r-1}(U, \mathbf{C}(\eta)) \otimes H^{r-1}(F^{r-1}, \mathbf{C})(\chi_1)^{\otimes(-1)}$$

as Hodge structures. Moreover, $H_c^n(U, \mathbf{C}(\eta))$ *is pure of weight* n.

Proof. Under the condition, we have

$$H_c^{i+r-1}(\tilde{\mathcal{Y}}, \mathbf{C})(\chi_1, \chi_2) \simeq H_c^i(U, \mathbf{C}(\eta)) \otimes H^{r-1}(F^{r-1}, \mathbf{C})(\chi_1)$$

We rewrite

$$u_1^m f_1 + \cdots + u_r^m f_r - 1 = u_1^m(f_1 + (\frac{u_2}{u_1})^m f_2 + \cdots + (\frac{u_r}{u_1})^m f_r) - 1$$

and

$$u_1^{a_1}\cdots u_r^{a_r}t_1^{b_1}\cdots t_n^{b_n}=u_1^{a_1+\cdots+a_r}(\frac{u_2}{u_1})^{a_2}\cdots(\frac{u_r}{u_1})^{a_r}t_1^{b_1}\cdots t_n^{b_n}.$$

Now we introduce a set of new variables $s_2=\frac{u_2}{r_1},\ldots,s_r=\frac{u_r}{u_1}$. Then we apply the non resonance condition for $f=f_1+s_2f_2+\cdots+s_rf_r$ and $(\alpha_1+\cdots+\alpha_r,\alpha_2,\ldots,\alpha_r,\beta_1,\ldots,\beta_n)$. The Newton polygon $\Delta(f)$ of f in $\mathbf{R}^{r-1}\times\mathbf{R}^n$ is the convex hull of $(0,\ldots,0,\Delta_1),(1,0,\ldots,0,\Delta_2),(0,\ldots,0,1,\Delta_r)$. Therefore the non resonance condition for $(\alpha_1+\cdots+\alpha_r,\alpha_2,\ldots,\alpha_r,\beta_1,\ldots,\beta_n)$ with respect to $\Delta(f)$ is expressed as

$$\hat{\Delta}\cap(\mathbf{Z}^{r+n}+(\alpha_1+\cdots+\alpha_r,\alpha_2,\ldots,\alpha_r,\beta_1,\ldots,\beta_n))=\emptyset,$$

where $\hat{\Delta}$ is the convex hull of $(1,\Delta(f))$ and 0. It is easy to see that $\hat{\Delta}$ is the convex hull of $(1,0,\ldots,0,\Delta_1),(1,1,0,\ldots,0,\Delta_2),\ldots,(1,0,\ldots,0,1,\Delta_r)$ and 0. Bye the linear change of base, $(p_1,\ldots p_r,q_1,\ldots q_n)\mapsto(p_1-p_2\cdots-p_r,p_2,\ldots,p_r,q_1,\ldots,q_n)$, $\hat{\Delta}$ maps onto $\tilde{\Delta}$ and $(\alpha_1+\cdots+\alpha_r,\ldots,\alpha_r,\beta_1,\ldots,\beta_n))$ maps to $(\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_n)$. As a consequence, it is equivalent to the resonance condition for $\Delta_1,\ldots,\Delta_r$ and $(\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_n)$.

Remark. In physics, the dummy variables $s_2,\ldots,s_r$ are called ghost field.

Definition (Mixed twisted Ehrhart polynomial). Let $E(\Delta_1,\ldots,\Delta_r,\alpha,\beta,t)$ be the formal power series defined by

$$E(\Delta_i,\alpha,\beta,t)=[\sum_{k=0}^{\infty}\#\{k\tilde{\Delta}\cap(\mathbf{Z}^{r+n}+(\alpha,\beta))\}](1-t)^{r+n-1}.$$

In the case where $r=1$, we have the following theorem.

Theorem 4.

(1) *(α_i) is non-resonant if and only if $l(j\Delta,\alpha_i)=l^*(j\Delta,\alpha_i)$ for all j.*
(2) *If (α_i) is non-resonant, $H^i(Z_F,\mathbf{C})(\alpha_i)$ is of pure of weight n and the twisted Ehrhart polynomial coincides with*

$$(-t)^n(\sum_{i=0}^{n}\dim Gr_F^iH^n(Z_f,\mathbf{C})(\alpha_i)t^{-i}).$$

Proof. This is a twisted version of the theorem due to Danilov and Khovanski. For the detail of the proof, see [T] or the paper in preparation.

We return to the case where r is general. Since

$$\#\{k\hat{\Delta}\cap(\mathbf{Z}^{r+n}+(\alpha_1+\cdots+\alpha_r,\alpha_r,\ldots,\alpha_r,\beta))\}=\#\{k\tilde{\Delta}\cap(\mathbf{Z}^{r+n}+(\alpha,\beta))\},$$

$E(\Delta_1,\ldots,\Delta_r,\alpha,\beta,t)$ turns out to be a polynomial by Theorem 4. It is called the mixed twisted Ehrhart polynomial. By Theorem 3 and Theorem 4, we have the following theorem.

Theorem 5. *Assume the condition of Theorem 3. Let $H^{p,q}$ be the (p,q)-component of the Hodge decomposition of $H_c^n(U, \mathbf{C}(\eta))$. Then we have*

$$\sum_{p=0}^{n} \dim(H^{p,q}) t^p = E(\Delta_i, \alpha, \beta, t) t^{-e},$$

where $e = <\alpha_1> + \cdots + <\alpha_r> - <\alpha_1 + \cdots + \alpha_r>$.

Remark. $E(\Delta_i, \alpha, \beta, 1)$ is equal to the Minkowski's mixed volume of $\{\Delta_i\}_i$.

Reference

[T] ワークショップ. 超曲面の特異点, 基本群, 有限被覆写像 報告集, P116〜118. 1995年. 東工大.